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1983 J. Phys. A: Math. Gen. 16 2479

(http://iopscience.iop.org/0305-4470/16/11/018)

J. Phys. A: Math. Gen. 16 (1983) 2479–2495. Printed in Great Britain

On phase separation in systems with continuous symmetry

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Received 13 October 1982

Abstract. It is proved that the interface in the isotropic D-vector model and its $D \to \infty$ limit, both with Kac–Helfand interactions, is diffuse at all temperatures. The interface does not stabilise even when a pinning potential of Abraham type is accommodated. The magnetisation profile is explicitly calculated and the interface width is shown to be proportional to the sample thickness.

1. Introduction

Establishing the existence of non-translationally invariant Gibbs states describing sharp interfaces is an interesting and non-trivial problem in the theory of phase transitions. It is known that the ferromagnetic Ising model in two dimensions has no such states (Gallavotti 1972, Aizenman 1980), while for three dimensions and more the contrary is true (Dobrushin 1972, van Beijeren 1975). The absence of a sharp interface in the two-dimensional Ising model is due to the existence of large fluctuations in the system, which make the two phases—when brought into contact—spread one over the other to a thickness proportional to $L^{1/2-\varepsilon}$ (L is the interface length), resulting in zero magnetisation profile (Gallavotti 1972, Abraham and Reed 1976). On the other hand, the fluctuations could destabilise the interface in the three-dimensional Ising model and thus a roughening transition at $T_R < T_c$ (3) has been conjectured (Burton *et al* 1951, Weeks *et al* 1973). However, the only models for which a roughening transition has been established rigorously are the SOS model (Frohlich and Spencer 1981) and models with a pinning potential of the sort studied by Abraham (1980). Thus, thermal fluctuations play an extremely important role in the phase separation and it is well known that they are controlled by the symmetry of the Hamiltonian as well as the lattice dimension and the range of the potential. For systems with continuous symmetry the fluctuations are expected to increase, and there is a phenomenological argument (Kittel 1971) according to which the interface should have a diverging width. We address ourselves in this paper to disproving the existence of a sharp interface for isotropic D-vector models ($D \geqslant 2$) and their spherical limit. In order to suppress the fluctuations and thus favour the localisation of the interface, we considered interactions of Kac–Helfand (1963) type. Moreover, we try to pin the interface near one boundary, by lowering the coupling there, as done by Abraham (1980) for the two-dimensional Ising model. In spite of this we found that for all temperatures the interface is not localised, even near the distorted boundary; its width is of the order of the thickness of the sample on the top and bottom of which we imposed 'mixed' boundary conditions. In this respect we have explicitly calculated the magnetisation profile, thus taking full advantage of the simplification induced by the long-range character of the interactions.

We think that our results substantiate the phenomenological prediction; for the non-existence of a sharp interface in a system where thermal fluctuations are damped as much as in the molecular field model should make very unlikely the appearance of a sharp interface in the corresponding system with short-range interactions. We would like also to note that the results obtained for the D-vector model hold even in the spherical limit and have been previously announced in a letter by Angelescu *et al* (1981a).

The method we devised to prove the mentioned results relies on establishing a certain isomorphism (very likely holding only when long-range interactions are used) between the magnetisation profile of the D-vector model and that of a 'D-vectorial spherical model'. The latter becomes in turn, provided appropriate limits are taken, the spherical model considered by Angelescu *et al* (1981a) and therefore there will be no need to study separately the spherical limit of the D-vector model.

The isotropic D-vector model with Kac–Helfand interactions, which will be our concern, can be described as follows. Consider a slab consisting of M copies of a rectangular array $\Lambda \subset \mathbb{Z}^{d-1}$ of 'spins'; the energy of a configuration $\{\boldsymbol{S}_{ir} \in \mathbb{R}^D \mid \|\boldsymbol{S}_{ir}\|^2 = D; r \in \Lambda, 1 \leq i \leq M\}$ is taken as

$$\mathscr{H}^{(\gamma)}_{M,\Lambda}(\{\boldsymbol{S}\}) = -(\gamma^{d-1}/2) \sum_{r,r' \in \Lambda} \rho(\gamma|r-r'|) \sum_{i,j=1}^{M} J_{ij} \boldsymbol{S}_{ir} \boldsymbol{S}_{jr'} - \sum_{r \in \Lambda} \sum_{i=1}^{M} D^{1/2} \boldsymbol{h}_i \boldsymbol{S}_{ir} \tag{1.1}$$

where $\rho : \mathbb{R}^{d-1} \to \mathbb{R}$ is a positive definite function such that $\int \rho(x)\, \mathrm{d}x = 1$, the scaling factor $\gamma > 0$ controls the interaction range,

$$J_{ij} = \tau \delta_{ij} + \delta_{|i-j|,1}, \qquad i,j = 1, \ldots, M \qquad (\tau \geq 2), \tag{1.2}$$

and $D^{1/2} \boldsymbol{h}_i \in \mathbb{R}^D$ is a homogeneous magnetic field acting on the ith layer. To describe the phase separation we shall eventually take all $\boldsymbol{h}_i = \boldsymbol{0}$, except for $\boldsymbol{h}_1$ and $\boldsymbol{h}_M$ in terms of which we describe the boundary conditions. Namely, consider the spins in two extremal layers $i = 0$ and $i = M+1$ fixed along two different directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, both vectors in $\mathbb{R}^D$; moreover, allow for a different coupling $J_{0,1} < 1$ at one boundary, and thus $\boldsymbol{h}_1$ and $\boldsymbol{h}_M$ are determined as $\boldsymbol{h}_1 = J_{01} \boldsymbol{e}_1$, $\boldsymbol{h}_M = J_{M,M+1} \boldsymbol{e}_2$, $\|\boldsymbol{h}_1\| = J_{01}$, $\|\boldsymbol{h}_M\| = 1$.

The model under consideration is the limit as $\gamma \downarrow 0$ of the model defined by the Hamiltonian (1.1) in the thermodyamic limit $\Lambda \to \infty$, and it is an inhomogeneous mean field model with M D-vector order parameters. In particular, the $\gamma \downarrow 0$ limit of the free energy per spin and per spin component exists by standard arguments (Thompson and Silver 1973) and is given by the absolute minimum with respect to $\{\boldsymbol{\xi}\} = (\boldsymbol{\xi}_i \mid \boldsymbol{\xi}_i \in \mathbb{R}^D, 1 \leq i \leq M)$ of the function

$$M f(\beta, \{\boldsymbol{h}\}, \{\boldsymbol{\xi}\}) = \tfrac{1}{2} \sum_{i,j=1}^{M} J_{ij} \boldsymbol{\xi}_i \boldsymbol{\xi}_j - \beta^{-1} \sum_{i=1}^{M} \mathscr{F}\left(\beta \left\| \sum_{j=1}^{M} J_{ij} \boldsymbol{\xi}_j + \boldsymbol{h}_i \right\|\right) \tag{1.3}$$

where

$$\mathscr{F}(\|\boldsymbol{x}\|) = D^{-1} \log \int_{\|\boldsymbol{S}\|^2 = D} \mathrm{d}\boldsymbol{S} \exp(D^{1/2} \boldsymbol{x} \cdot \boldsymbol{S}) \tag{1.4}$$

is the free energy of one spin in the external magnetic field $D^{1/2}\boldsymbol{x}$ and has the properties (i)–(iv) listed in appendix 1. Taking into account that $\mathscr{F}$ has linear behaviour at infinity ($|\mathscr{F}'| < 1$) and that the matrix J introduced by (1.2) is strictly positive definite, one concludes that $f(\beta,\{\boldsymbol{h}\},\cdot)$ attains its minimum at finite distance. Since $\mathscr{F}$ is an even function, $f(\beta,\{\boldsymbol{h}\},\cdot)$ is differentiable on $\mathbb{R}^{DM}$ and hence its minimum points are among its stationary points, i.e. among the solutions of the system

$$\boldsymbol{\xi}_i=\mathscr{F}'\left(\beta\left\|\sum_{j=1}^{M}J_{ij}\boldsymbol{\xi}_j+\boldsymbol{h}_i\right\|\right)\cdot\left(\sum_{j=1}^{M}J_{ij}\boldsymbol{\xi}_j+\boldsymbol{h}_i\right)\Big/\left\|\sum_{j=1}^{M}J_{ij}\boldsymbol{\xi}_j+\boldsymbol{h}_i\right\|,\qquad 1\leqslant i\leqslant M. \tag{1.5}$$

The minimum point $\{\boldsymbol{\xi}\}$ is intimately related to the magnetisation profile as shown in § 2. The techniques required for solving (1.5) are developed in § 2 and two appendices. The lemma on convex functions in appendix 2 seems to be new and therefore of independent interest. Section 3 contains the main result disproving the existence of the sharp interface in the D-vector model ($D\geqslant 2$), while § 4 indicates how to use this proof for the generalised spherical model considered in Angelescu *et al* (1981a).

2. The layer magnetisations and the minimum point

We have seen in § 1 that the model under consideration has a mean field character and thus solving it requires finding the absolute minimum of the function $f(\beta,\{\boldsymbol{h}\},\cdot)$ defined by (1.3). We are, however, interested in the phase separation phenomenon, which requires studying the magnetisation profile across slab thickness. This is equivalent to the detailed characterisation of the point at which the absolute minimum of f is attained. To be more precise, suppose $f(\beta,\{\boldsymbol{h}\},\cdot)$ attains the absolute minimum at a unique point $\boldsymbol{\xi}(\{\boldsymbol{h}\})$, where moreover the Hessian matrix $\partial^2 f/\partial\xi_{i\alpha}\,\partial\xi_{j\beta}$ is non-singular; then the layer magnetisations at the given β and $\boldsymbol{h}$,

$$\boldsymbol{m}_i=\lim_{\gamma\downarrow 0}\lim_{\Lambda\to\infty}\left\langle D^{-1/2}|\Lambda|^{-1}\sum_{\mu\in\Lambda}\boldsymbol{S}_{\mu i}\right\rangle_{\gamma,\Lambda}^{(M)}=-\lim_{\gamma\downarrow 0}\lim_{\Lambda\to\infty}M\nabla_{\boldsymbol{h}}f_{\gamma,\Lambda}^{(M)}(\beta,\{\boldsymbol{h}\}), \tag{2.1}$$

are nothing but $\boldsymbol{m}_i=\boldsymbol{\xi}_i(\{\boldsymbol{h}\})$. ($\langle\ \rangle_{\gamma,\Lambda}^{(M)}$ and $f_{\gamma,\Lambda}^{(M)}$ denote respectively the Gibbs state and free energy defined by the Hamiltonian (1.1).) Indeed, the minimum is attained on a solution of the system (1.5). Since the Hessian matrix is non-singular, for all $\{\boldsymbol{h}'\}$ in a neighbourhood of $\{\boldsymbol{h}\}$ the system (1.5) has a unique solution $\boldsymbol{\xi}(\{\boldsymbol{h}'\})$ in the neighbourhood of $\boldsymbol{\xi}(\{\boldsymbol{h}\})$, which depends differentiably on $\{\boldsymbol{h}'\}$ and is the unique point of absolute minimum of $f(\beta,\{\boldsymbol{h}'\},\cdot)$. (For the latter fact, remark that the minimum point is always in the compact $\|\boldsymbol{\xi}_i\|\leqslant 1$, $i=1,\ldots,M$, as seen from (1.5).) Thus, $f(\beta,\{\boldsymbol{h}'\},\boldsymbol{\xi}(\{\boldsymbol{h}'\}))$ is differentiable at $\{\boldsymbol{h}'\}=\{\boldsymbol{h}\}$. Remembering that $f_{\gamma,\Lambda}^{(M)}(\beta,\{\boldsymbol{h}'\})$ are convex functions of $\{\boldsymbol{h}'\}$ and converge for $\Lambda\to\infty$, $\gamma\downarrow 0$ to $f(\beta,\{\boldsymbol{h}'\},\boldsymbol{\xi}(\{\boldsymbol{h}'\}))$, the assertion follows from Griffiths' theorem (Griffiths 1964).

In proposition 2.1 we shall exhibit a convenient domain for $\{\boldsymbol{h}\}$ on which the situation above takes place and suited for describing the phase separation. We start with a few definitions. Let us fix $\boldsymbol{e}\in\mathbb{R}^D$ and define

$$\mathscr{D}_{\boldsymbol{e}}=\{\boldsymbol{x}\in\mathbb{R}^D\,|\,\boldsymbol{x}\cdot\boldsymbol{e}>0\},\qquad \mathscr{U}_{\boldsymbol{e}}=\left\{\{\boldsymbol{x}\}=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_M)\in\bar{\mathscr{D}}_{\boldsymbol{e}}^M\,\middle|\,\sum_{i=1}^{M}\boldsymbol{x}_i\cdot\boldsymbol{e}>0\right\}, \tag{2.2}$$

where $\bar{\mathscr{D}}$ stands for the closure of $\mathscr{D}$. For $\{\boldsymbol{h}^*\}=(\boldsymbol{h}_1^*,\ldots,\boldsymbol{h}_M^*)\in\bar{\mathscr{U}}_{\boldsymbol{e}}$, with $\boldsymbol{h}_i^*\boldsymbol{e}=0$, we define

$$\mathscr{E}_{\boldsymbol{h}^*}=\left\{\{\boldsymbol{h}\}\,\middle|\,\boldsymbol{h}_i=\sum_{j=1}^{M}a_{ij}\boldsymbol{h}_j^*+a_i\boldsymbol{e},\ 1\leqslant i\leqslant M,\ a_{ij}\in\mathbb{R},\ a_i\geqslant 0\right\}. \tag{2.3}$$

Proposition 2.1. Let $\{\boldsymbol{h}\}\in\mathscr{U}_e$ and $\beta>0$ be given. Then the absolute minimum of $f(\beta,\{\boldsymbol{h}\},\cdot)$ defined by (1.3) is attained at one and only one point, $\boldsymbol{\xi}(\{\boldsymbol{h}\})$. Moreover:

(i) $\boldsymbol{\xi}(\{\boldsymbol{h}\})\in\mathscr{D}_e^M$ and is the unique solution in $\mathscr{D}_e^M$ of the system (1.5);

(ii) $\boldsymbol{\xi}(\{\boldsymbol{h}\})$ is differentiable on $\mathscr{U}_e$;

(iii) there exists $\lim\boldsymbol{\xi}(\{\boldsymbol{h}\})$ for $\{\boldsymbol{h}\}\to\{\boldsymbol{h}^*\}$, $\{\boldsymbol{h}\}\in\mathscr{E}_{\boldsymbol{h}^*}\cap\mathscr{U}_e$.

Proof. The proof proceeds in several steps:

(a) the points of absolute minimum of f are in $\mathscr{D}_e^M$;

(b) the system (1.5) has in $\mathscr{D}_e^M$ one and only one solution;

(c) the Hessian matrix of f is non-degenerate at $\boldsymbol{\xi}=\boldsymbol{\xi}(\{\boldsymbol{h}\})$;

(d) the existence of the limit in (iii).

(a) For any $\{\boldsymbol{h}\}\in\mathscr{U}_e$ and $\{\boldsymbol{\xi}\}\in\mathbb{R}^{MD}$ one can write

$$\boldsymbol{h}_i=\boldsymbol{h}_i'+a_i\boldsymbol{e},\qquad \boldsymbol{\xi}_i=\boldsymbol{\xi}_i'+\alpha_i\boldsymbol{e},\qquad i=1,2,\ldots,M,\tag{2.4}$$

where $\boldsymbol{h}_i'\cdot\boldsymbol{e}=\boldsymbol{\xi}_i'\cdot\boldsymbol{e}=0$; obviously $a_i\geqslant 0$, $1\leqslant i\leqslant M$, and $\Sigma_{i=1}^M a_i>0$. Let us denote by $\boldsymbol{a},\boldsymbol{\alpha}\in\mathbb{R}^M$ the vectors whose components are a_i, α_i respectively.

The function $f(\beta,\{\boldsymbol{h}\},\{\boldsymbol{\xi}\})$ defined by (1.3) can accordingly be written as

$$Mf(\beta,\{\boldsymbol{h}\},\{\boldsymbol{\xi}\})=K(\{\boldsymbol{\xi}'\})+g(\{\boldsymbol{h}'\},\{\boldsymbol{\xi}'\};\boldsymbol{a},\boldsymbol{\alpha})$$

where

$$K(\{\boldsymbol{\xi}'\})=\tfrac{1}{2}\sum_{i,j=1}^{M}J_{ij}\boldsymbol{\xi}_i'\cdot\boldsymbol{\xi}_j',\qquad g(\{\boldsymbol{h}'\},\{\boldsymbol{\xi}'\};\boldsymbol{a},\boldsymbol{\alpha})=\tfrac{1}{2}\sum_{i,j=1}^{M}J_{ij}\alpha_i\alpha_j-\sum_{i=1}^{M}\phi_i((J\boldsymbol{\alpha}+\boldsymbol{a})_i)$$

the functions $\phi_i(x)$ being introduced by the relations

$$\phi_i(x)=\beta^{-1}\mathscr{F}(\beta(K_i^2+x^2)^{1/2}),\qquad K_i=\|(J\boldsymbol{\xi}'+\boldsymbol{h}')_i\|,\qquad i=1,2,\ldots,M.\tag{2.5}$$

The set $\{\phi_i\}_{1\leqslant i\leqslant M}$ satisfies the properties (i)–(iv) considered in appendix 1 and hence the lemma stated there can be applied in order to see that $\inf_{\boldsymbol{\alpha}}g(\{\boldsymbol{h}'\},\{\boldsymbol{\xi}'\};\boldsymbol{a},\boldsymbol{\alpha})$ is realised at one and only one point $\boldsymbol{\alpha}(\{\boldsymbol{\xi}'\})>\boldsymbol{0}$. Considering now a point $\{\boldsymbol{\xi}^*\}$ at which $f(\beta,\{\boldsymbol{h}\},\cdot)$ attains the absolute minimum, it is obvious that $\inf_{\boldsymbol{\alpha}}g(\{\boldsymbol{h}'\},\{\boldsymbol{\xi}^{*\prime}\};\boldsymbol{a},\boldsymbol{\alpha})$ is attained at $\boldsymbol{\alpha}^*$, where $\{\boldsymbol{\xi}^{*\prime}\}$ and $\{\boldsymbol{\alpha}^*\}$ represent the decomposition of $\{\boldsymbol{\xi}^*\}$, cf (2.4). It follows that $\boldsymbol{\alpha}^*>\boldsymbol{0}$, i.e. $\{\boldsymbol{\xi}^*\}\in\mathscr{D}_e^M$.

(b) We can restrict from now on the domain of all the functions entering the system (1.5) to the set $\mathscr{L}=\{\boldsymbol{\xi}\in\mathscr{D}_e^M\,|\,\xi_i\equiv\|\boldsymbol{\xi}_i\|<1,\ 1\leqslant i\leqslant M\}$, and define $\boldsymbol{\Psi}\colon\mathscr{L}\to\mathbb{R}^M$ by

$$\Psi_i(\{\boldsymbol{\xi}\})=\beta^{-1}\mathscr{F}'^{-1}(\xi_i)/\xi_i,\qquad i=1,2,\ldots,M.\tag{2.6a}$$

Taking into account that, for any solution $\boldsymbol{\xi}_i$ of (1.5), $\xi_i=\mathscr{F}'(\beta\|(J\boldsymbol{\xi}+\boldsymbol{h})_i\|)$, the system (1.5) is equivalent on $\mathscr{L}$ with

$$[\operatorname{diag}\boldsymbol{\Psi}(\{\boldsymbol{\xi}\})-J]\boldsymbol{\xi}=\boldsymbol{h}\tag{2.6b}$$

where diag $\boldsymbol{\gamma}$ denotes the $M\times M$ diagonal matrix whose elements are the components of the vector $\boldsymbol{\gamma}$. Now, if $\{\boldsymbol{\xi}^*\}\in\mathscr{L}$ satisfies (2.6b), the matrix $[\operatorname{diag}\boldsymbol{\Psi}(\{\boldsymbol{\xi}^*\})-J]$ transforms the strictly positive vector $\boldsymbol{\alpha}^*$ into the positive vector $\boldsymbol{a}\neq\boldsymbol{0}$ ($\boldsymbol{\alpha}^*$ and $\boldsymbol{a}$ are defined by the decomposition (2.4)); thus $[\operatorname{diag}\boldsymbol{\Psi}(\{\boldsymbol{\xi}^*\})-J]$ is strictly positive definite (see e.g. Angelescu *et al* 1979). Then (2.6b) implies

$$\xi_i^{*2}=[(\operatorname{diag}\boldsymbol{\Psi}(\{\boldsymbol{\xi}^*\})-J)^{-1}\boldsymbol{h}]_i^2,\qquad i=1,2,\ldots,M.\tag{2.7}$$

In order to prove that (2.6b) has exactly one solution in $\mathscr{L}$, we shall try to find a change of variables under which (2.6b) transforms into the extremum condition for a certain strictly convex differentiable function. It will be useful to introduce the function $G: [0, 1) \to R$ defined by

$$G(x^2) = \beta^{-1} \mathscr{F}'^{-1}(x)/x \tag{2.8}$$

and remark that

$$\Psi_i(\{\xi\}) = G((\xi_i^2)), \qquad i = 1, 2, \ldots, M. \tag{2.9}$$

Keeping in mind the properties of $\mathscr{F}$ (listed in appendix 1 under the conditions (i)–(iv)), one can see that G is strictly increasing, continuous and transforms $[0, 1)$ onto $[\beta^{-1}, \infty)$. Besides, the function G is differentiable and $(G^{-1})' > 0$ on (β^{-1}, ∞). If H is a primitive of G^{-1} it can be defined on $[\beta^{-1}, \infty)$ where moreover it is strictly convex. Let $\mathscr{D}$ be the open and convex set

$$\mathscr{D} = \{\gamma \in \mathbb{R}^M \,|\, \text{diag}\, \gamma - J > 0, \gamma_i > \beta^{-1}\} \tag{2.10}$$

and let $T_h : \mathscr{D} \to R$ be the function

$$T_h(\gamma) = \sum_{i,j=1}^{M} (\text{diag}\, \gamma - J)_{ij}^{-1} \boldsymbol{h}_i \cdot \boldsymbol{h}_j + \sum_{i=1}^{M} H(\gamma_i) \tag{2.11}$$

where $\{h\} \in \mathscr{U}_e$. Making use of the fact that the mapping $X \to X^{-1}$ is convex on the set of strictly positive definite matrices (Lieb and Ruskai 1974), and recalling that H is strictly convex, it results that T_h is strictly convex on $\mathscr{D}$. Hence the system

$$\partial T_h/\partial \gamma_i = G^{-1}(\gamma_i) - [(\text{diag}\, \gamma - J)^{-1} \boldsymbol{h}]_i^2 = 0, \qquad i = 1, 2, \ldots, M, \tag{2.12}$$

has at most one solution on $\mathscr{D}$.

Let $\{\xi^*\}$ be a solution of (2.6*b*) and let $\gamma^* \equiv \Psi(\{\xi^*\})$. Then γ^* is a stationary point of T_h. Indeed, we have already seen that $[\text{diag}\, \Psi(\{\xi^*\}) - J] > 0$. On the other hand $\Psi_i(\{\xi^*\}) = G(\xi_i^{*2}) \geq G(\alpha_i^{*2}) > \beta^{-1}$ and thus $\gamma^* \in \mathscr{D}$. Taking into account that by the definition of γ^*, $\xi_i^{*2} = G^{-1}(\gamma_i^*)$ and invoking (2.7), we find that γ^* satisfies (2.12). But T_h has only one stationary point and therefore $\{\xi^*\} \in \Psi^{-1}(\gamma^*)$. Further, we shall consider another solution $\{\xi^{**}\}$ and note that necessarily $\Psi(\{\xi^{**}\}) = \gamma^*$. Then (2.6*b*) will provide $\xi^{**} = (\text{diag}\, \gamma^* - J)^{-1} \boldsymbol{h} = \xi^*$.

(c) In order to see that the Hessian matrix of f is non-degenerate at the point $\xi = \xi(\{h\})$ one can consider its expression

$$\frac{M \partial^2 f}{\partial \xi_{i\mu}\, \partial \xi_{j\nu}} = J_{ij} - \sum_{p=1}^{M} J_{ip} \eta_p^{-1} \mathscr{F}'(\beta \eta_p) J_{pj}$$
$$+ \beta \sum_{p=1}^{M} J_{ip} \eta_p^{-2} \eta_{p\mu} \eta_{p\nu} (\beta^{-1} \eta_p^{-1} \mathscr{F}'(\beta \eta_p) - \mathscr{F}''(\beta \eta_p)) J_{pj}, \tag{2.13}$$

where $\boldsymbol{\eta}_p = (J\xi + \boldsymbol{h})_p$ and $\eta_p = \|\boldsymbol{\eta}_p\|$. The last term on the RHS of (2.13) defines a matrix of the form JAJ and since $\mathscr{F}'(x)/x > \mathscr{F}''(x)$ for $x \geq 0$ it can be seen that A is positive definite and so is JAJ. It remains to check that the remaining part is strictly positive definite for $\xi = \xi(\{h\})$. But $\mathscr{F}'(\beta \eta_p)/\eta_p = 1/\Psi_p(\xi(\{h\}))$ when $\xi = \xi(\{h\})$ and since $\text{diag}\, \Psi(\xi(\{h\})) > J > 0$, it can be easily seen that indeed the first two terms in (2.13) define a strictly positive matrix.

The proof of statement (ii) in proposition 2.1 has thus been completed.

(d) We shall begin by noting that T_h introduced by (2.11) is well defined on $\mathscr{D}$ for all $\{h\} \in \mathbb{R}^{MD}$. We shall denote by $\underline{T}_h$ its lower semicontinuous extension to $\bar{\mathscr{D}}$ which is strictly convex on $\bar{\mathscr{D}}$ (i.e. strictly convex on the set on which $\underline{T}_h$ is finite) and consequently it has only one point $\gamma(\{h\})$ at which its absolute minimum is attained. When $\{h\}\to\{h^*\}$, $T_h\to T_{h^*}$ uniformly on compacts in $\mathscr{D}$ and we can apply the lemma in appendix 2 to see that $\gamma(\{h\})\to\gamma(\{h^*\})$.

Let us consider now $\{h^*\}\in\bar{\mathscr{U}}_e$, $h_i^*e=0$, $1\le i\le M$, and let $e_1, e_2, \ldots, e_p$ be an orthonormal basis in the subspace of $\mathbb{R}^D$ generated by h_i^*, $1\le i\le M$. If $\{h\}\in\mathscr{E}_{h^*}$:

$$h_i = h_i' + a_i e, \qquad h_i' = \sum_{\mu=1}^{p} h_{i\mu}' e_\mu, \qquad a_i \ge 0, 1\le i\le M. \tag{2.14}$$

If moreover $\{h\}\in\mathscr{E}_{h^*}\cap\mathscr{U}_e$, $\sum_{i=1}^M a_i>0$. Note also that $p\le M$.

As $\gamma(\{h\})$ converges when $\{h\}\to\{h^*\}$, $\{h\}\in\mathscr{E}_{h^*}\cap\mathscr{U}_e$, every limit point ξ^* of $\xi(\{h\})$ will satisfy

$$[\operatorname{diag}\gamma(\{h^*\})-J]\alpha^*=0, \qquad [\operatorname{diag}\gamma(\{h^*\})-J]\xi_\mu^{*\prime}=h_\mu^*, \qquad 1\le\mu\le p, \tag{2.15}$$

$$\gamma_i(\{h^*\})=G(\alpha_i^{*2}+\xi_i^{*\prime 2}), \qquad 1\le i\le M, \tag{2.16}$$

where $h_\mu^*=(h_{i\mu}^*: 1\le i\le M)$; $\xi_i^{*\prime}=\Sigma_{\mu=1}^p \xi_{i\mu}^{*\prime}e_\mu$, α^* realise the decomposition (2.14) of ξ^*, while $\xi_\mu^{*\prime}=(\xi_{i\mu}^{*\prime}: 1\le i\le M)$.

The proof will consist in showing that (2.15) and (2.16) determine ξ^* uniquely in terms of h^*. If $[\operatorname{diag}\gamma(\{h^*\})-J]>0$, then (2.15) provides ξ^* uniquely. If, however, this matrix has the zero eigenvalue (necessarily simple with normalised eigenvector $v>0$), then $p<M$, and $h_\mu^*\cdot v=0$, $1\le\mu\le p$. Under these conditions, (2.15) shows that $\alpha^*=\eta v$ ($\eta\ge 0$) and $\xi_\mu^{*\prime}=\lambda_\mu v+u_\mu$, with u_μ ($u_\mu\cdot v=0$) uniquely determined and linearly independent. To compute η and λ_μ, use is made of (2.16) written in the form

$$G^{-1}(\gamma_i(\{h^*\}))=\left(\eta^2+\sum_{\mu=1}^p\lambda_\mu^2\right)v_i^2+2\sum_{\mu=1}^p\lambda_\mu u_{i\mu}v_i+\sum_{\mu=1}^p u_{i\mu}^2, \qquad 1\le i\le M. \tag{2.17}$$

Summing over i and using $u_\mu v=0$, one gets $\eta^2+\Sigma_{\mu=1}^p\lambda_\mu^2$; then (2.17) becomes a linear system of rank p, which determines λ_μ, $1\le\mu\le p$.

This completes the proof of proposition 2.1.

3. The properties of the magnetisation profile

It has been shown so far that whenever the layer magnetic fields h_i are all lying in a half-space (conventionally fixed by e) the Gibbs state of the considered physical system can be essentially determined. Thus we have seen that calculating the layer magnetisations m_i when $\{h\}\in\mathscr{U}_e$ is equivalent to finding the unique solution in $\mathscr{D}_e^M$ of the system (1.5). Moreover, it has been shown that whenever a certain limiting procedure (closely resembling that through which the usual spontaneous magnetisation is defined) is adopted, one can determine the magnetisations m_i even when $\{h\}$ lies on the boundary of $\mathscr{U}_e$. The importance of this point stems from the fact that in the phase separation problem which is our concern here, we have to consider exactly the case when $\{h\}$ lies on the boundary of $\mathscr{U}_e$ or, more specifically, when h_1 and h_M have opposite directions while the other magnetic fields are zero. The study of the phase separation in our model is thus reduced to the study of the properties of the unique solution in $\mathscr{D}_e^M$ of the system (1.5) ($\{h\}\in\mathscr{U}_e$), as well as to the study of this solution's limit when $\{h\}$ approaches in a certain way the boundary of $\mathscr{U}_e$. More precisely, one has to take only $h_1\equiv\xi_0$ and $h_M\equiv\xi_{M+1}$ different from zero to account for the boundary conditions as shown in § 1; inverting $\mathscr{F}'$ in (1.5) and introducing the function

$$F(x)=\beta^{-1}\mathscr{F}'^{-1}(x)-\tau x, \tag{3.1}$$

one is left with the system

$$\boldsymbol{\xi}_{i+1}+\boldsymbol{\xi}_{i-1}=(\boldsymbol{\xi}_i/\xi_i)F(\xi_i), \qquad \boldsymbol{\xi}_i\in\mathscr{D}_{\boldsymbol{e}},\ 1\leq i\leq M, \tag{3.2}$$

with the boundary conditions

$$\boldsymbol{\xi}_0,\ \boldsymbol{\xi}_{M+1}\in\bar{\mathscr{D}}_{\boldsymbol{e}}, \qquad \theta\equiv\measuredangle(\boldsymbol{\xi}_0,\boldsymbol{\xi}_{M+1})<\pi, \qquad 0<\xi_0,\ \xi_{M+1}\leq 1, \tag{3.3}$$

where $\xi_i\equiv\|\boldsymbol{\xi}_i\|$. From now on we shall study the properties of the unique solution of (3.2), (3.3) and of their limit as $\theta\to\pi$ (see § 2).

Before proceeding any further some simple remarks will be in order. Thus, since $\boldsymbol{\xi}_i\cdot\boldsymbol{e}>0$, $1\leq i\leq M$, and $\boldsymbol{\xi}_0\cdot\boldsymbol{e}$, $\boldsymbol{\xi}_{M+1}\cdot\boldsymbol{e}\geq 0$, equation (3.2) implies

$$F(\xi_i)>0, \qquad 1\leq i\leq M. \tag{3.4}$$

Therefore choosing as positive the sense of rotation of the vector $\boldsymbol{\xi}_0$ over $\boldsymbol{\xi}_{M+1}$ of angle less than π, all the angles $\theta_i\equiv\measuredangle(\boldsymbol{\xi}_i,\boldsymbol{\xi}_{i+1})$ will satisfy

$$\theta_i\in[0,\pi), \qquad 0\leq i\leq M, \qquad 0\leq\sum_{i=0}^{M}\theta_i=\theta<\pi. \tag{3.5}$$

Now, it is easy to see that (3.2) can be written as

$$\begin{aligned}&\xi_{i+1}\cos\theta_i+\xi_{i-1}\cos\theta_{i-1}=F(\xi_i),\\ &\xi_{i+1}\sin\theta_i=\xi_{i-1}\sin\theta_{i-1},\end{aligned} \qquad i=1,2,\ldots,M, \tag{3.6}$$

whence

$$\xi_{i+1}\xi_i\sin\theta_i=\xi_1\xi_0\sin\theta_0\equiv c, \qquad i=1,2,\ldots,M. \tag{3.7}$$

Let us write down some properties which we shall need in what follows.

(i) The quantity c defined in (3.7) satisfies the inequality

$$0<(M+1)c\leq\theta<\pi. \tag{3.8}$$

(ii) If the solution of (3.2) is such that $\theta_i\in[0,\pi/2]$ for $0\leq i\leq M$, then it satisfies the relation

$$(\xi_{i+1}^2-c^2/\xi_i^2)^{1/2}+(\xi_{i-1}^2-c^2/\xi_i^2)^{1/2}=F(\xi_i), \qquad i=1,2,\ldots,M. \tag{3.9}$$

(iii) Let

$$g(c;x)\equiv F(x)-2(x^2-c^2/x^2)^{1/2} \tag{3.10}$$

be defined on $0\leq\sqrt{c}\leq x<1$. The function $g(c,\cdot)$ is convex on its domain. If

$$\beta_c\equiv(\tau+2)^{-1}, \tag{3.11}$$

then for $\beta\leq\beta_c$ the equation $g(c,\xi)=0$ has only one solution $\xi=c=0$; $g(c,\xi)$ is strictly positive otherwise. For $\beta>\beta_c$ the sign of the function g and its zeros cannot be simply expressed analytically and figures 1 and 2 will provide the missing analysis.

(iv) Let $\{\boldsymbol{\xi}_i\}_{i\leq i\leq M}$ be a solution of (3.2) such that $\theta_i\in[0,\pi/2]$, $0\leq i\leq M$. If for $1\leq i_0\leq M$, ξ_{i_0} is a local minimum (i.e. $\xi_{i_0\pm1}\geq\xi_{i_0}$) with $\xi_{i_0}\geq\sqrt{c}$, then either $g(c,\xi_{i_0})>0$ or $g(c,\xi_{i_0})=0$ in which case $\xi_i=\xi_{i_0}$, $0\leq i\leq M+1$. If ξ_{i_0} is a local maximum (i.e. $\xi_{i_0\pm1}\leq\xi_{i_0}$) then either $g(c,\xi_{i_0})<0$ or $g(c,\xi_{i_0})=0$ in which case $\xi_i=\xi_{i_0}$, $0\leq i\leq M+1$.

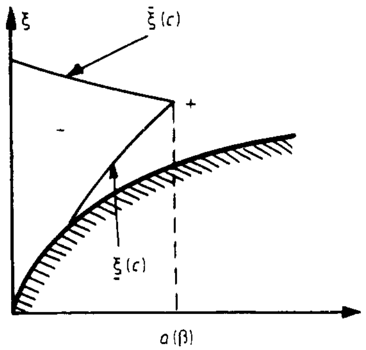

Figure 1. The sign of the function $g(c, \xi)$ for $\beta \in (\beta_c, \tau^{-1}]$.

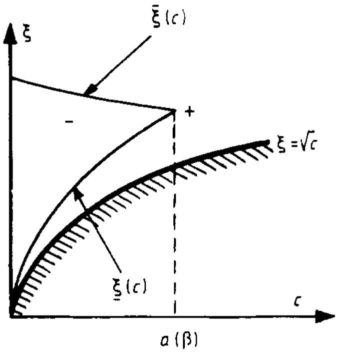

Figure 2. The sign of the function $g(c, \xi)$ for $\beta > \tau^{-1}$.

(v) Let $\beta > \beta_c$ and define the continuous function

$$\underline{\underline{\xi}}(c) \equiv \max\{\sqrt{c}, \underline{\xi}(c)\}, \qquad c \in [0, a(\beta)], \tag{3.12}$$

where $\underline{\xi}(c)$, $\bar{\xi}(c)$ and $a(\beta)$ have been introduced in figures 1 and 2 (here $\underline{\xi}(c)$ is taken equal to zero when not defined). Now let $\beta > \beta_c$ and the boundary conditions $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_{M+1}$ be fixed as specified in (3.3); then the solution of the system (3.2) satisfies, for M large enough (depending on β, ξ_0 and ξ_{M+1}), the relations

$$\begin{aligned} &\xi_i > \underline{\underline{\xi}}(c), \qquad && i = 1, 2, \ldots, M, \\ &\theta_i \in [0, \pi/2), && i = 0, 1, \ldots, M. \end{aligned} \tag{3.13}$$

Indeed, let us introduce the continuous functions $\boldsymbol{\xi}_0(t), \boldsymbol{\xi}_{M+1}(t) \in \mathscr{D}_e$ defined on $t \in [0, 1]$ such that for any $t \in [0, 1]$, $\xi_0(t) = \xi_0$, $\xi_{M+1}(t) = \xi_{M+1}$ and $\boldsymbol{\xi}_0(0) = \xi_0 \boldsymbol{e}$, $\boldsymbol{\xi}_{M+1}(0) = \xi_{M+1}\boldsymbol{e}$ while $\boldsymbol{\xi}_0(1) = \boldsymbol{\xi}_0$, $\boldsymbol{\xi}_{M+1}(1) = \boldsymbol{\xi}_{M+1}$. These functions have been chosen such that for any $t \in [0, 1]$ the system (3.2) with boundary conditions $\boldsymbol{\xi}_0(t)$, $\boldsymbol{\xi}_{M+1}(t)$ has a unique solution $\{\boldsymbol{\xi}_i(t)\}_{1 \leq i \leq M}$ which depends continuously on t. Then $\cos\theta_i(t)$ depends continuously on t and so does $\theta_i(t)$ ($\theta_i(t) \in [0, \pi)$). Let us now choose M so large that the inequality $\min\{a(\beta), \xi_0^2, \xi_{M+1}^2\} > \pi/(M+1)$ is satisfied. Recalling (i), one will then have $c(t) < \min\{a(\beta), \xi_0^2, \xi_{M+1}^2\}$. Taking into account now that $\underline{\xi}(c)$ is continuous and $\underline{\xi}(0) = 0$, we can find M_0 such that the inequalities $c(t) < \min\{a(\beta), \xi_0^2, \xi_{M+1}^2\}$ and $\underline{\underline{\xi}}(c(t)) < \min\{\xi_0, \xi_{M+1}\}$ hold simultaneously for $M \geq M_0$. Consider from now on $M \geq M_0$. At $t = 0$ the angle θ between $\boldsymbol{\xi}_0(0)$ and $\boldsymbol{\xi}_{M+1}(0)$ is zero and (3.5) imposes $\theta_i(0) = 0$, $0 \leq i \leq M$, hence $c(0) = 0$ and as $\underline{\xi}(0) = 0$ equation (3.13) is fulfilled. Suppose that there exists $t_0 \in [0, 1]$ such that (3.13) is true; then using the above-mentioned continuities, (3.13) will be satisfied in a certain neighbourhood in $[0, 1]$ of t_0. Therefore the set of points $t \in [0, 1]$ for which (3.13) is obeyed is open in $[0, 1]$. If on the other hand $\{\boldsymbol{\xi}_i(t)\}_{1 \leq i \leq M}$ do not satisfy (3.13) for any $t \in [0, 1]$, then there exists $t_0 \in (0, 1]$ and $1 \leq i_0 \leq M$ such that for any $t \in [0, t_0)$, $\{\boldsymbol{\xi}_i(t)\}_{1 \leq i \leq M}$ obey the relations (3.13), while for $t = t_0$ either

$$\text{(A)} \qquad \begin{aligned} &\xi_i(t_0) \geq \xi_{i_0}(t_0) = \underline{\underline{\xi}}(c(t_0)), \qquad && i = 1, 2, \ldots, M, \\ &\theta_i(t_0) \in [0, \pi/2), && i = 0, 1, \ldots, M, \end{aligned}$$

is valid, or

$$\text{(B)} \qquad \begin{aligned} &\xi_i(t_0) > \underline{\underline{\xi}}(c(t_0)), \qquad && i = 1, 2, \ldots, M, \\ &\theta_i(t_0) \leq \theta_{i_0}(t_0) = \pi/2, && i = 0, 1, \ldots, M. \end{aligned}$$

Suppose (A) holds; then since $c(t_0) < a(\beta)$ we shall have $\xi_i(t_0) \geq \xi_{i_0}(t_0) = \underline{\underline{\xi}}(c(t_0)) \geq \sqrt{c(t_0)}$ and (iv) can be applied, providing either $g(c(t_0), \xi_{i_0}(t_0)) > 0$ and hence $\xi_{i_0}(t_0) > \bar{\xi}(c(t_0)) > \underline{\underline{\xi}}(c(t_0))$ contradicting (A), or $\xi_i(t_0) = \xi_{i_0}(t_0)$, $i = 0, 1, \ldots, M+1$, in which case $\underline{\underline{\xi}}(c(\bar{t}_0)) = \xi_{i_0}(t_0) = \xi_0(t_0) = \xi_0$ contradicting the choice of M_0. Let us examine therefore the case (B). Since $M \geq M_0$, it results that $\xi_0, \xi_{M+1} > \sqrt{c(t_0)}$ and $\xi_i(t_0) > \underline{\underline{\xi}}(c(t_0)) \geq \sqrt{c(t_0)}$, $1 \leq i \leq M$; hence $\xi_i(t_0) > \sqrt{c(t_0)}$, $0 \leq i \leq M+1$, whence $\xi_{i_0+1}(t_0)\xi_{i_0}(t_0) > c(t_0)$. But (B) implies $c(t_0) = \xi_{i_0+1}(t_0)\xi_{i_0}(t_0)$. $\sin\theta_{i_0}(t_0) = \xi_{i_0+1}(t_0)\xi_{i_0}(t_0)$ and thus we have arrived at a contradiction.

(vi) Let $\beta > 0$ and ξ_0 and ξ_{M+1} be fixed. If $\{\xi_i\}_{1 \leq i \leq M}$ is the solution of (3.2), then for M large enough (depending on β, ξ_0, ξ_{M+1}) there exists $i_0 = i_0(M) \in \{0, 1, \ldots, M+1\}$ such that $\xi_i = \xi_{i_0+i-i_0}$ is a monotonic function of $|i - i_0|$. Indeed, for $\beta \in (0, \beta_c]$ we shall have $F(\xi) \geq 2\xi$ and (3.6) leads to

$$\xi_{i+1} + \xi_{i-1} \geq 2\xi_i, \qquad i = 1, 2, \ldots, M. \tag{3.14}$$

Now it can be easily seen that the solution cannot have a local maximum in ξ_{i_0}, $1 \leq i_0 \leq M$. Then $\{\xi_i\}_{0 \leq i \leq M+1}$ is either a monotonic sequence (in which case $i_0(M) = 0$ or $M+1$) or it has a unique local minimum which is attained in i_0 (or two neighbouring points), $1 \leq i_0 \leq M$, in which case $i_0(M) = i_0$. Suppose that $\beta > \beta_c$ and take M_0 as given in (v). Then for $M \geq M_0$ one has $\xi_i > \underline{\underline{\xi}}(c)$, $1 \leq i \leq M$, and $\theta_i \in [0, \pi/2)$, $0 \leq i \leq M$. Therefore if for $j_0 \in \{1, 2, \ldots, M\}$ the sequence $\{\xi_i\}_{0 \leq i \leq M+1}$ has a local minimum, one applies (iv) ($\xi_i > \underline{\underline{\xi}}(c) \geq \sqrt{c}$, $\theta_i \in [0, \pi/2)$) to arrive at $g(c, \xi_{j_0}) > 0$ whence

$$\xi_{j_0} > \bar{\xi}(c). \tag{3.15a}$$

Analogously if for $i_0 \in \{1, 2, \ldots, M\}$ the solution has a local maximum, (iv) can again be invoked to establish that

$$\xi_{i_0} < \bar{\xi}(c), \tag{3.15b}$$

which together with (3.15*a*) shows that the sequence $\{\xi_i\}_{0 \leq i \leq M+1}$ cannot have both a local minimum and maximum. Thus $\{\xi_i\}_{0 \leq i \leq M+1}$ is either monotonic (in which case $i_0(M) = 0$ or $M+1$) or it has a unique local minimum (or maximum) which is attained in $i_0 \in \{1, 2, \ldots, M\}$; in this case $i_0(M) = i_0$.

Having established the properties (i)–(vi) we can pass to proving the following proposition.

Proposition 3.1. Let $\beta \neq \beta_c$ and $\{\xi_i\}_{1 \leq i \leq M}$ be the solution of the system (3.2) with boundary conditions ξ_0 and ξ_{M+1}. Then there exist $b > 0$, $B > 0$ and M_0 all depending on β, ξ_0, ξ_{M+1} such that for $M \geq M_0$

$$|\bar{\xi}(c) - \xi_i| < B\,\mathrm{e}^{-bd_i}, \qquad i = 0, 1, \ldots, M+1, \tag{3.16}$$

where $d_i = \min\{i, M+1-i\}$.

Proof. Let M be so large that (vi) holds. We shall consider first that $\beta \in (0, \beta_c)$. In this case

$$x^{-1}(F(x) - 2x) \geq F'(0) - 2 \equiv \omega^{-1} > 0 \qquad \text{for } x \geq 0 \tag{3.17}$$

and recalling (3.6), $F(\xi_i) \leq \xi_{i+1} + \xi_{i-1}$ $(1 \leq i \leq M)$ whence

$$\xi_i \leq \omega(\xi_{i+1} + \xi_{i-1} - 2\xi_i), \qquad i = 1, 2, \ldots, M. \tag{3.18}$$

Further, for $i \neq i_0 = i_0(M)$ given by (vi), (3.18) leads to

$$\begin{aligned} &\xi_i \leqslant [\omega/(1+\omega)]^i \xi_0, && i \leqslant i_0 - 1, \\ &\xi_i \leqslant [\omega/(1+\omega)]^{M+1-i} \xi_{M+1}, && i \geqslant i_0 + 1. \end{aligned} \tag{3.19}$$

For $i = i_0$, (3.18) implies $\xi_{i_0} \leqslant (\xi_{i_0+i} + \xi_{i_0-i})[\omega/(1+2\omega)]$ whence one gets

$$\xi_{i_0} \leqslant \{[\omega/(1+\omega)]^{i_0}\xi_0 + [\omega/(1+\omega)]^{M+1-i_0}\xi_{M+1}\}[(1+\omega)/(1+2\omega)]. \tag{3.20}$$

Taking $b = \ln(1+1/\omega)$, $B = 2\max\{\xi_0, \xi_{M+1}\}$ and taking into account that $\bar{\xi}(c) = 0$ for $\beta \in (0, \beta_c)$ one obtains (3.16).

Let us suppose now that $\beta > \beta_c$. Then the sequence $\{\xi_i\}_{0 \leqslant i \leqslant M+1}$ has by property (vi) either only one local extremum in $i_0 \in \{1, 2, \ldots, M\}$ or it is monotonic.

(a) If $\{\xi_i\}_{0 \leqslant i \leqslant M+1}$ has a local minimum in i_0 then $\xi_i = \xi_{i_0+i-i_0}$ is monotonically increasing with $|i - i_0|$. Besides, (3.15*a*) will provide

$$\max\{\xi_0, \xi_{M+1}\} \equiv \xi^* \geqslant \xi_i > \bar{\xi}(c), \qquad i = 0, 1, \ldots, M+1. \tag{3.21}$$

Note now that for $\beta > \beta_c$, $\bar{\xi}(c) > 0$ and recall that $g(c, x)$ is convex on $x \geqslant \sqrt{c}$; denoting $g'(c, \bar{\xi}(c))$ by $1/\tilde{\omega}(c) > 0$, it is easy to see that

$$\xi - \bar{\xi}(c) \leqslant \tilde{\omega}(c) g(c, \xi) \tag{3.22}$$

as soon as $\xi \geqslant \bar{\xi}(c)$. We apply this inequality to $\xi = \xi_i$.

(b) If $\{\xi_i\}_{0 \leqslant i \leqslant M+1}$ has a local maximum in i_0 then ξ_i is monotonically decreasing with $|i - i_0|$; (3.15*b*) will provide

$$\min\{\xi_0, \xi_{M+1}\} \equiv \xi_* \leqslant \xi_i < \bar{\xi}(c), \qquad i = 0, 1, \ldots, M+1. \tag{3.23}$$

Recalling the choice of M in (vi) we shall have $\xi_* > \sqrt{c}$ and again the convexity of g provides

$$\bar{\xi}(c) - \xi \leqslant -\tilde{\omega}(c) g(c, \xi) \tag{3.24}$$

where $\tilde{\omega}(c) = -g(c, \xi_*)(\bar{\xi}(c) - \xi_*)^{-1}$ and $\xi < \bar{\xi}(c)$. The inequality (3.24) leads similarly, when applied to $\xi = \xi_i$, to the inequality (3.16).

(c) The case of monotonic $\{\xi_i\}_{0 \leqslant i \leqslant M+1}$ can be easily reduced to (a) or (b) above.

Proposition 3.2. Let $\{\boldsymbol{\xi}_i\}_{1 \leqslant i \leqslant M}$ be the solution of the system (3.2) with fixed boundary conditions $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_{M+1}$. Let $\beta > \beta_c$, $\theta_i = \measuredangle(\boldsymbol{\xi}_i, \boldsymbol{\xi}_{i+1})$, $\theta = \measuredangle(\boldsymbol{\xi}_0, \boldsymbol{\xi}_{M+1})$ and d_i, b, B be the quantities introduced in proposition 3.1. Then there exist $C > 0$ and M_0 depending on β, ξ_0, ξ_{M+1} only, such that for $M \geqslant M_0$

$$|\theta_i - \theta/(M+1)| < CM^{-2} \ln M \tag{3.25}$$

whenever i is such that $d_i > (1/b) \ln BM$, while otherwise

$$\theta_i = O(1/M). \tag{3.26}$$

Before proving this proposition we shall give without proof the following lemma.

Lemma 3.3. Let $\varphi_i \in (0, \pi/2)$ and $\Sigma_{i=1}^N \varphi_i = \varphi$. If there exist $A > 0$, $a > 0$ and $i_0 \in \{1, 2, \ldots, N\}$ such that

$$a \sin \varphi_{i_0} \leqslant \sin \varphi_i \leqslant A \sin \varphi_{i_0}, \qquad i = 1, 2, \ldots, N, \tag{3.27}$$

then

$$(a\varphi/AN)[1-(A\varphi/aN)^2]^{1/2} \leqslant \sin\varphi_i \leqslant A\varphi/aN, \qquad i=1,2,\ldots,N. \tag{3.28}$$

Proof of proposition 3.2. If $\theta=0$, (3.25) is trivially satisfied and therefore we will consider $\theta\in(0,\pi)$. Let us choose M_1 such that for $M\geqslant M_1$, proposition 3.1 together with the properties (v), (vi) hold true. The property (vi) and (3.15) imply

$$\begin{aligned} &\xi_i \geqslant \min\{\xi_0, \xi_{M+1}, \bar{\xi}(c)\} \equiv \xi_*(c), \\ &\xi_i \leqslant \max\{\xi_0, \xi_{M+1}, \bar{\xi}(c)\} \equiv \xi^*(c), \end{aligned} \qquad i=0,1,\ldots,M+1, \tag{3.29}$$

while (3.7) leads to

$$\sin\theta_i = (\xi_{i_0+1}\xi_{i_0}/\xi_{i+1}\xi_i)\sin\theta_{i_0}, \qquad i, i_0\in\{0,1,\ldots,M\}, \tag{3.30}$$

providing

$$(\xi_*(c)/\xi^*(c))^2\sin\theta_{i_0} \leqslant \sin\theta_i \leqslant (\xi^*(c)/\xi_*(c))^2\sin\theta_{i_0}.$$

Thus we can apply the above lemma and find that there exist two positive constants C^* and C_* both M-independent such that

$$C_*[\theta/(M+1)] \leqslant \theta_i \leqslant C^*[\theta/(M+1)], \qquad i=0,1,\ldots,M. \tag{3.31}$$

Now let $S_M=\{i\,|\,d_i\geqslant(1/b)\ln BM, 0\leqslant i\leqslant M+1\}$ and let $\tilde{M}$ be its cardinal. S_M is non-void for M large enough, and

$$(M-\tilde{M})/M = O(M^{-1}\ln M). \tag{3.32}$$

Besides, recalling proposition 3.1, we shall have

$$|\xi_i-\bar{\xi}(c)|<1/M \tag{3.33}$$

whenever $i\in S_M$. Equations (3.30), (3.33) imply

$$\begin{aligned} &A^{-1}\sin\theta_{i_0} \leqslant \sin\theta_i \leqslant A\sin\theta_{i_0}, \qquad i, i_0\in S_M, \\ &A=(\bar{\xi}(c)+1/M)^2(\bar{\xi}(c)-1/M)^{-2}. \end{aligned} \tag{3.34}$$

But since $\bar{\xi}(c)-\bar{\xi}(0)=O(c)$ and $c\leqslant\pi/(M+1)$, one gets

$$A=1+O(1/M). \tag{3.35}$$

For θ_i, $i\in S_M$, lemma 3.3 provides

$$\begin{aligned} &A^{-2}\tilde{M}^{-1}\tilde{\theta}[1-(A^2\tilde{\theta}/\tilde{M})^2] \leqslant \sin\theta_i \leqslant \tilde{\theta}A^2\tilde{M}^{-1}, \qquad i\in S_M, \\ &\tilde{\theta}\equiv\sum_{i\in S_M}\theta_i. \end{aligned} \tag{3.36}$$

But

$$\theta=\tilde{\theta}+\sum_{i\notin S_M}\theta_i \leqslant \tilde{\theta}+C^*(M-\tilde{M})/(M+1)=\tilde{\theta}+O(M^{-1}\ln M),$$

whence

$$0<\theta-\tilde{\theta}=O(M^{-1}\ln M). \tag{3.37}$$

Collecting now the estimations (3.32), (3.35), (3.37) and using the inequality (3.36) we obtain

$$\theta_i=\theta/M+O(M^{-2}\ln M), \qquad i\in S_M. \tag{3.38}$$

Proposition 3.4. Let $\{\boldsymbol{\xi}_i\}_{1\le i\le M}$ be the solution of the system (3.2) with the boundary conditions fixed as follows: $\boldsymbol{\xi}_0, \boldsymbol{\xi}_{M+1} \in \bar{\mathscr{D}}_{\boldsymbol{e}}$, $\boldsymbol{\xi}_0 \cdot \boldsymbol{e} = 0$, $\boldsymbol{\xi}_{M+1} \cdot \boldsymbol{e} > 0$. Then for $\beta > \beta_c$

$$\lim_{M\to\infty, k/M\to x} \boldsymbol{\xi}_k = \bar{\xi}(0)[(\boldsymbol{\xi}_0/\xi_0)\cos\theta x + \boldsymbol{e}\sin\theta x] \qquad (3.39)$$

where $\theta = \measuredangle(\boldsymbol{\xi}_0, \boldsymbol{\xi}_{M+1})$, $x \in (0, 1)$.

Proof. Note that $\boldsymbol{\xi}_k = \xi_k[(\boldsymbol{\xi}_0/\xi_0)\cos(\Sigma_{i=0}^k \theta_i) + \boldsymbol{e}\sin(\Sigma_{i=0}^k \theta_i)]$. Proposition 3.1 ensures that $\xi_k \to \bar{\xi}(0)$ whenever $x \in (0, 1)$ while proposition 3.2 enables us to assert that

$$\lim_{M\to\infty, k/M\to x} \sum_{i=0}^{k} \theta_i = \theta x$$

whence (3.39) results.

4. Concluding remarks

Our result, proposition 3.3, shows that looking at a region far away from both boundaries, the state is translationally invariant. The direction of the local order parameter is intermediate between the directions of the boundary fields and depends on how the thermodynamic limit is taken. Also, proposition 3.2 provides the following information on the behaviour near boundaries:

$$\lim_{M\to\infty} \boldsymbol{\xi}_k = (\boldsymbol{\xi}_0/\xi_0) m_k(\beta, \xi_0).$$

Here $m_k(\beta, \xi_0) > 0$ are the layer magnetisations of a semi-infinite system with boundary field ξ_0 (as defined in Angelescu *et al* 1981b) which approach exponentially fast the bulk spontaneous magnetisation. In other words, the layers at finite distance from one boundary, however small the coupling to it, do not feel the phase at the other boundary. Then we can conclude that an interface cannot be localised either deep in the bulk or near a boundary. The same conclusion holds also in the spherical limit of the model we considered here as announced in Angelescu *et al* (1981a). To see this it is sufficient to remark that the function whose minimum is sought in the spherical model is nothing but the limit when $\{\boldsymbol{h}\}$ converges to the boundary of $\mathscr{U}_{\boldsymbol{e}}$ of the function $T_{\boldsymbol{h}}(\boldsymbol{\gamma})$ appearing here (see (2.11)) as an artifact of the proof of proposition 2.1. Thus the D-vector model and the spherical one can both be solved in one stroke.

The interface problem has been recently studied for short-range interactions by Abraham and Robert (1980) within the spherical model of Berlin and Kac (1952). They found as well that the interface is diffuse at all temperatures. However, the magnetisation profile there has some unphysical features, which led them to suggest that the model itself is inadequate for considering such 'non-translationally invariant' problems. Our generalised spherical model is free of this objection and indeed the profile we obtain is physically sound.

Acknowledgments

The authors would like to thank Professors H Wagner and C J Thompson for many interesting discussions on phase separation in magnets.

Appendix 1

Let $\{\phi_i\}_{i=1,2,\ldots,M}$ be real functions defined on $\mathbb{R}$ satisfying for every $1 \leqslant i \leqslant M$ the following conditions:

(i) $\phi_i(x) = \phi_i(-x)$, $\phi_i(\mathbb{R}) \subset \mathbb{R}_+$ and $\phi_i \in C^3(\mathbb{R})$,

(ii) $\phi_i'(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \phi_i'(x) = 1$,

(iii) $\phi_i''(x) > 0$ for every $x \in \mathbb{R}$,

(iv) $\phi_i'''(x) < 0$ for $x > 0$.

Let us now introduce the function $\mathscr{F}_\phi : \mathbb{R}^M \to \mathbb{R}$ defined by

$$\mathscr{F}_\phi(\boldsymbol{x}) = \tfrac{1}{2}(J\boldsymbol{x}, \boldsymbol{x}) - \beta^{-1} \sum_{i=1}^{M} \phi_i((J\boldsymbol{x} + \boldsymbol{h})_i) \tag{A1.1}$$

where J is an $M \times M$ strictly positive definite real matrix, positive (with respect to the componentwise order in $\mathbb{R}^M$) and irreducible while $\boldsymbol{h} \in \mathbb{R}^M$, $\boldsymbol{h} \neq \boldsymbol{0}$ and $\boldsymbol{h} \geqslant \boldsymbol{0}$. Then:

Lemma. The absolute minimum of the function $\mathscr{F}_\phi$ on $\mathbb{R}^M$ is attained in one and only one point $\boldsymbol{x}^{(\phi)}$ satisfying $\boldsymbol{x}^{(\phi)} > \boldsymbol{0}$. Moreover, if $\{\tilde{\phi}_i\}_{i=1,2,\ldots,M}$ is another set of functions satisfying the conditions (i)–(iv) and $\phi_i' \geqslant \tilde{\phi}_i'$, $i = 1, 2, \ldots, M$, then $\boldsymbol{x}^{(\phi)} \geqslant \boldsymbol{x}^{(\tilde{\phi})}$.

Proof. Let $\boldsymbol{\varphi} : \mathbb{R}^M \to \mathbb{R}^M$ be a function defined by

$$\varphi_i(\boldsymbol{x}) = \phi_i'(\beta(J\boldsymbol{x} + \boldsymbol{h})_i), \qquad i = 1, 2, \ldots, M. \tag{A1.2}$$

Now, since J is strictly positive definite and ϕ_i have a linear behaviour at infinity (see (ii)) it results that $\mathscr{F}_\phi$ attains its absolute minimum in at least one point which should be among the solutions of the system

$$\nabla_x \mathscr{F}_\phi = J\boldsymbol{x} - J^{\mathrm{T}} \boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{0}.$$

But $J = J^{\mathrm{T}}$ and J^{-1} exists and hence this system can be brought into the form

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{x}) \tag{A1.3}$$

showing that the stationary points of $\mathscr{F}_\phi$ are the fixed points of $\boldsymbol{\varphi}$.

Let us note that the application $\boldsymbol{\varphi}$ has the following properties.

(a) ϕ is monotonically increasing on $\mathbb{R}^M$ (with respect to the order introduced above); its fixed points $\boldsymbol{x}$ satisfy $|x_i| < 1$.

(b) If $\boldsymbol{x} \leqslant \boldsymbol{\varphi}(\boldsymbol{x})$ (or $\boldsymbol{x} \geqslant \boldsymbol{\varphi}(\boldsymbol{x})$) the sequnce $\boldsymbol{\varphi}^{0n}(\boldsymbol{x})$ converges when $n \to \infty$ to a fixed point of $\boldsymbol{\varphi}$.

(c) There exists $n \in N$ such that $\boldsymbol{\varphi}^{0n}(\{\boldsymbol{x} \mid \boldsymbol{x} \geqslant \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}\}) \subset \{\boldsymbol{x} \mid \boldsymbol{x} > \boldsymbol{0}\}$. In particular if $\boldsymbol{x} \geqslant \boldsymbol{0}$, $\boldsymbol{x} \neq \boldsymbol{0}$ is a fixed point of $\boldsymbol{\varphi}$ then $\boldsymbol{x} > \boldsymbol{0}$.

(d) $\boldsymbol{\varphi}$ has one and only one fixed point in the set $\{\boldsymbol{x} \mid \boldsymbol{x} \geqslant \boldsymbol{0}, \boldsymbol{x} \neq \boldsymbol{0}\}$, $\boldsymbol{\xi} = \lim_{n\to\infty} \boldsymbol{\varphi}^{0n}(\boldsymbol{0})$.

Indeed, the application $\boldsymbol{x} \to J\boldsymbol{x}$ is increasing on $\mathbb{R}^M$ and $\phi_i''(x) > 0$ for $x \in \mathbb{R}$, whence (a) results easily.

(b) Note that $\boldsymbol{\varphi}(\mathbb{R}^M) = (-1, 1)^M$ and account for (a) before.

(c) Since J is positive and irreducible there exists $n \in N$ large enough such that $(J^n)_{\alpha\beta} > 0$, for any $\alpha, \beta \in \{1, 2, \ldots, M\}$. Besides for $\boldsymbol{y} \geqslant \boldsymbol{0}$, $y_i > 0$ and $J_{ki} > 0$ one has $\varphi_k(\boldsymbol{y}) > 0$.

(d) Account for ϕ_i' being strictly concave on $x \geq 0$ to arrive at

$$\varphi_i(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda\varphi_i(\boldsymbol{x}) + (1-\lambda)\varphi_i(\boldsymbol{y}), \qquad i = 1, 2, \ldots, M, \tag{A1.4}$$

with at least one strict inequality; here $\boldsymbol{x}, \boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{x} \neq \boldsymbol{y}$ and $\lambda \in (0, 1)$.

Let $\boldsymbol{x}, \boldsymbol{y} \in \{\boldsymbol{z} \mid \boldsymbol{z} \geq 0\}$ be two distinct fixed points of $\boldsymbol{\varphi}$; (c) then provides $\boldsymbol{x}, \boldsymbol{y} > \boldsymbol{0}$. As $\boldsymbol{x} \neq \boldsymbol{y}$ one can find $\lambda_0 \notin [0, 1]$ and an index $i_0 \in \{1, 2, \ldots, M\}$ such that

$$\boldsymbol{z} = \lambda_0 \boldsymbol{x} + (1-\lambda_0)\boldsymbol{y} \geq \boldsymbol{0} \qquad \text{and} \qquad z_{i_0} = \lambda_0 x_{i_0} + (1-\lambda_0) y_{i_0} = 0.$$

With (A1.4) we shall have $\boldsymbol{\varphi}(\boldsymbol{z}) \leq \boldsymbol{z}$ and $\boldsymbol{\varphi}(\boldsymbol{z}) \neq \boldsymbol{z}$. Property (a) above leads to $\boldsymbol{\varphi}^{0n}(\boldsymbol{z}) \leq \boldsymbol{z}$ for any n, while (c) enables us to find n such that $\boldsymbol{\varphi}^{0n}(\boldsymbol{z}) > \boldsymbol{0}$ and thus $\boldsymbol{z} > \boldsymbol{0}$, which contradicts $z_{i_0} = 0$.

We are now prepared to prove the lemma. We shall begin by noting that if $\boldsymbol{y}$ is an arbitrary fixed point of $\boldsymbol{\varphi}$ then there exists a fixed point $\boldsymbol{\xi} \geq 0$ such that $|y_i| \leq \xi_i$, $1 \leq i \leq M$. Indeed, let $\boldsymbol{y}^*$ be the vector whose components $y_i^* = |y_i|$. Then one has $\boldsymbol{0} \leq \boldsymbol{y}^* \leq \boldsymbol{\varphi}(\boldsymbol{y}^*)$ (note that $\boldsymbol{h} \geq 0$). Hence by (b) $\boldsymbol{\varphi}^{0n}(\boldsymbol{y}^*) \to \boldsymbol{\xi}$, monotonically increasing, $\boldsymbol{\xi}$ being a fixed point of $\boldsymbol{\varphi}$; thus $\boldsymbol{\varphi}^{0n}(\boldsymbol{y}^*) \leq \boldsymbol{\xi}$. Further, let $\boldsymbol{y}$ be an arbitrary stationary point of $\mathscr{F}_\phi$. Then

$$\mathscr{F}_\phi(\boldsymbol{y}) = \beta^{-1} \sum_{i=1}^{M} G_i(y_i) \tag{A1.5}$$

where

$$G_i(x) = (x/2)\phi_i'^{-1}(x) - \phi_i \circ \phi_i'^{-1}(x) - (\beta/2)h_i x \tag{A1.6}$$

which is strictly decreasing for $x > 0$ and has the property $G_i(x) \geq G_i(|x|)$. Using now (A1.5) we shall have $\mathscr{F}_\phi(\boldsymbol{y}) > \mathscr{F}_\phi(\boldsymbol{y}^*) > \mathscr{F}_\phi(\boldsymbol{\xi})$, where $\boldsymbol{\xi} = \lim_{n\to\infty} \boldsymbol{\varphi}^{0n}(\boldsymbol{y}^*)$. Thus the absolute minimum of $\mathscr{F}_\phi$ is attained at the only fixed point of $\boldsymbol{\varphi}$, $\boldsymbol{\xi}$.

Appendix 2

Let $D \subset \mathbb{R}^n$ be an open convex set and let $f: D \to R$ be a continuous convex function. We shall denote by $\underline{f}$ the extension of f at $\bar{D}$ defined by

$$\underline{f}(\boldsymbol{x}) = \varliminf_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ \boldsymbol{y}\in D}} f(\boldsymbol{y}), \qquad \boldsymbol{x} \in \bar{D}. \tag{A2.1}$$

Evidently f and $\underline{f}$ have the same lower bound and $\underline{f}$ is lower semicontinuous on $\bar{D}$. Let us also note that for every $a > \inf f$ the set $Q(\underline{f}, a) = \{\boldsymbol{x} \in \bar{D} \mid \underline{f}(\boldsymbol{x}) \leq a\}$ is convex and closed (Rockafellar 1970); we shall also remark that $\underline{f}$ attains its absolute minimum on the set $Q(f) = \bigcap_{a > \inf f} Q(\underline{f}, a) = \{\boldsymbol{x} \in \bar{D} \mid \underline{f}(\boldsymbol{x}) = \inf \underline{f}\}$.

Let $\mathscr{C}$ be the set of all continuous convex functions on D and $\mathscr{C}_0$ the set of all functions $f \in \mathscr{C}$ for which $Q(f)$ is non-void and bounded. Then the following continuity property holds.

Lemma. Let $f \in \mathscr{C}_0$. Then for every $\varepsilon > 0$ there exists $\eta > 0$ and a compact $K \subset D$ such that for every $g \in \mathscr{C}$ which satisfies

$$\sup_{\boldsymbol{x}\in K} |f(\boldsymbol{x}) - g(\boldsymbol{x})| < \eta$$

one has $g \in \mathscr{C}_0$ and

$$Q(g) \subset \{x \mid d(x, Q(f)) < \varepsilon\}, \qquad |\inf f - \inf g| < \varepsilon, \tag{A2.2}$$

where $d(x, Q(f))$ is the distance between x and $Q(f)$.

Proof. Let $f \in \mathscr{C}_0$ and $\varepsilon > 0$ be given; without restricting the generality one can suppose $\inf f = 0$. Let V be the set

$$V = \{x \mid d(x, Q(f)) < \varepsilon\}. \tag{A2.3}$$

Since $Q(f)$ is non-void, convex and compact, it results that V is convex and open and $\bar{V}$ compact. Moreover, as f is lower semicontinuous on $\bar{D}$ it can be asserted (Rockafellar 1970) that there is $\delta > 0$ such that

$$Q(f, \delta) \subset V.$$

Let us choose $\delta' \in (0, \delta)$ with $\delta' < \varepsilon/4$; taking into account (A2.1) we find that there exists $x_0 \in D$ such that

$$f(x_0) = \delta'. \tag{A2.4}$$

Let us consider now the set

$$K_\delta = \{z \mid f(z) = \delta, z \in [x_0, y], y \in (\bar{D} \setminus V) \cap \bar{V}\}. \tag{A2.5}$$

Then the following properties hold:

(i) $K_\delta \subset V \cap D$,

(ii) for every $y \in \bar{D} \setminus V$, $[x_0, y] \cap K_\delta \neq \varnothing$,

(iii) K_δ is a compact set.

Indeed:

(i) Obviously $K_\delta \subset Q(f, \delta) \subset V$. On the other hand if $z \in K_\delta$ then $z \in [x_0, y]$ with $y \in \bar{D} \setminus V$; but $y \in \bar{D} \setminus V$ implies $f(y) > \delta$ and since $x_0 \in D$ then $[x_0, y] \subset D$ and therefore $K_\delta \subset D$.

(ii) Let $y \in \bar{D} \setminus V$; as $x_0 \in V \cap D$ and V is open and convex one has $[x_0, y] \cap V = [x_0, y_1]$, $[x_0, y] \cap (\bar{D} \setminus V) = [y_1, y]$, where $y_1 \in \bar{V} \cap (\bar{D} \setminus V)$.

Evidently $f|[y_1, y] > \delta$ and therefore supposing that $[x_0, y] \cap K_\delta = \varnothing$ implies necessarily that $\delta \notin f([x_0, y_1))$. But since f is continuous on D and $[x_0, y_1) \subset D$, then $f([x_0, y_1))$ is an interval and since $f(x_0) = \delta' < \delta$ we have $f|[x_0, y_1) < \delta$. Recalling that f is lower semicontinuous, we have

$$f(y_1) = \varliminf_{\substack{z \to y_1 \\ z \in [x_0, y_1]}} f(z) \leqslant \delta$$

which contradicts $f(y_1) > \delta$. Hence $[x_0, y] \cap K_\delta \neq \varnothing$.

(iii) We already know that K_δ is a bounded set. It remains therefore to show that K_δ is a closed set. Let us consider a convergent sequence $\{z_n\} \subset K_\delta$, $z_n \to z$. As $z_n \in K_\delta$ there exist $y_n \in (\bar{D} \setminus V) \cap \bar{V}$ and $\lambda_n \in [0, 1]$ such that

$$z_n = (1 - \lambda_n)x_0 + \lambda_n y_n. \tag{A2.6}$$

But $(\bar{D} \setminus V) \cap \bar{V}$ is a compact set and hence there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to $y \in (\bar{D} \setminus V) \cap \bar{V}$. Equation (A2.6) provides that

$$|z_{n_k} - x_0| = \lambda_{n_k} |y_{n_k} - x_0|. \tag{A2.7}$$

Let us remark now that $\lim_k|y_{n_k}-x_0|>0$ (otherwise, $x_0=y$ which is impossible since $x_0\in V$ while $y\in\bar{D}\backslash V$), which in turn implies that λ_{n_k} is convergent. Let $\lambda=\lim_k\lambda_{n_k}$. Then equation (A2.6) provides that $z=(1-\lambda)x_0+\lambda y$, with $\lambda\in[0,1]$, $y\in(\bar{D}\backslash V)\cap V$. But since $\{z_n\}\subset K_\delta\subset Q(f,\delta)$ it results that $z\in Q(f,\delta)\cap V$ and since $z\in[x_0,y]$, we have $z\in[x_0,y]\cap V=[x_0,y)\subset D$. Then $z\in D$ and therefore is a continuity point of f which leads to

$$f(z)=\lim_k f(z_{n_k})=\delta$$

implying that $z\in K_\delta$. Hence the set K_δ is closed.

Let us proceed further and consider for $\alpha>1$ the set

$$K=\{z\mid z=x_0+\alpha^{-1}(x-x_0),\, x\in\bar{V}\cap\bar{D}\}. \tag{A2.8}$$

Obviously $K\subset V\subset D$ and for $\alpha\to1$, K is getting closer to $V\cap D$; since the compact $K_\delta\subset V\cap D$ there will exist $\alpha>1$, which can be safely taken less than 2 such that $K_\delta\subset K\subset V\cap D$. Consider now $g\in\mathscr{C}$ such that $\sup_{x\in K}|f(x)-g(x)|<\eta$ with $\eta<\frac{1}{4}\varepsilon$, $\frac{1}{2}(\delta-\delta')$ and remark that for $x\in\bar{D}\backslash V$, $x=\lambda x_1-(\lambda-1)x_0$ with $x_1\in K_\delta\subset K$ and $\lambda>1$. Therefore

$$\begin{aligned}g(x)&\geqslant\lambda g(x_1)-(\lambda-1)g(x_0)\\&\geqslant\lambda(f(x_1)-\eta)-(\lambda-1)(f(x_0)+\eta)=\delta-\eta+(\lambda-1)(\delta-\delta'-2\eta).\end{aligned}$$

Since $\delta-\delta'>2\eta$ we have

$$g(x)>\delta-\eta>\tfrac{1}{2}(\delta+\delta'). \tag{A2.9}$$

On the other hand

$$g(x_0)\leqslant f(x_0)+\eta=\delta'+\eta<\tfrac{1}{2}(\delta+\delta') \tag{A2.10}$$

which together with (A2.9) implies that inf g is attained on $\bar{D}\cap V$ and therefore $g\in\mathscr{C}_0$ and $Q(g)\subset V\cap\bar{D}$.

We shall consider now that $x\in\bar{V}\cap\bar{D}$ and with (A2.8) one can write

$$x=\alpha z-(\alpha-1)x_0,\qquad z,x_0\in K,$$

and $\alpha\in(1,2)$ as previously fixed. Using the convexity of g we obtain

$$g(x)\geqslant\alpha g(z)-(\alpha-1)g(x_0)\geqslant\alpha(f(z)-\eta)-(\alpha-1)(f(x_0)+\eta)$$

and since $f(z)\geqslant\inf f=0$ and $\alpha\in(1,2)$ we get

$$g(x)\geqslant-3\eta-\delta'\geqslant-\varepsilon.$$

As $Q(g)\subset\bar{V}\cap\bar{D}$ we have

$$\inf g\geqslant-\varepsilon. \tag{A2.11}$$

On the other hand $g(x_0)\leqslant f(x_0)+\eta=\delta'+\eta\leqslant\frac{1}{4}\varepsilon+\frac{1}{4}\varepsilon<\varepsilon$ whence

$$\inf g\leqslant\varepsilon \tag{A2.12}$$

which together with (A2.11) implies

$$|\inf g|\leqslant\varepsilon,$$

whence one gets the last part of our lemma.

Note added in proof. The information obtained in propositions 3.1 and 3.2 about the magnetisation profile allows the calculation of the helicity modulus (Fisher *et al* 1973), defined as

$$Y(\beta)=\lim_{M\to\infty}\frac{2M^2}{\theta^2}[f_{\theta,M}(\beta)-f_{0,M}(\beta)]$$

where $f_{\theta,M}(\beta)$ is the free energy for the slab with M layers under boundary conditions $\boldsymbol{\xi}_0$, $\boldsymbol{\xi}_{M+1}$, with $\theta=\measuredangle(\boldsymbol{\xi}_0,\boldsymbol{\xi}_{M+1})$. Remarking that for the solution of (3.2), $c=\theta\bar{\xi}(c)^2/(M+1)+O[(\ln M)/(M+1)^2]$ and that $\bar{\xi}(0)=\xi_B(\beta)=$ the bulk spontaneous magnetisation of the model, one obtains $Y(\beta)=\xi_B(\beta)^2$.

References

Abraham D B 1980 *Phys. Rev. Lett.* **44** 1165
Abraham D B and Reed P 1976 *Commun. Math. Phys.* **49** 35
Abraham D B and Robert M A 1980 *J. Phys. A: Math. Gen.* **13** 2229
Aizenman M 1980 *Commun. Math. Phys.* **73** 83
Angelescu N, Bundaru M and Costache G 1979 *J. Phys. A: Math. Gen.* **12** 2457
—— 1981a *J. Phys. A: Math. Gen.* **14** L533
Angelescu N, Bundaru M, Costache G and Nenciu G 1981b *J. Stat. Phys.* **24** 529
van Beijeren H 1975 *Commun. Math. Phys.* **40** 1
Berlin T H and Kac M 1952 *Phys. Rev.* **86** 821
Burton J A, Cabrera N and Frank F C 1951 *Phil. Trans. R. Soc.* A **243** 299
Dobrushin R L 1972 *Theory Probability Appl.* **17** 582
Fisher M E, Barber M N and Jasnow D 1973 *Phys. Rev.* A **8** 1111
Frohlich J and Spencer T 1981 *Commun. Math. Phys.* **81** 527
Gallavotti G 1972 *Commun. Math. Phys.* **27** 103
Griffiths R B 1964 *J. Math. Phys.* **4** 1078
Kac M and Helfand E 1963 *J. Math. Phys.* **4** 1078
Kittel C 1971 *Introduction to Solid State Physics* 4th edn (New York: Wiley)
Lieb E H and Ruskai M B 1974 *Adv. Math.* **12** 269
Rockafellar R T 1970 *Convex Analysis* (Princeton: Princeton UP)
Thompson C J and Silver H 1973 *Commun. Math. Phys.* **33** 53
Weeks J D, Gilmer G H and Leamy H J 1973 *Phys. Rev. Lett.* **31** 549